

Citation for published version:

Moinat, A & Weber, H 2020, 'Local bounds for stochastic reaction diffusion equations', *Electronic Journal of Probability*, vol. 25, pp. 1-26. <https://doi.org/10.1214/19-EJP397>

DOI:

[10.1214/19-EJP397](https://doi.org/10.1214/19-EJP397)

Publication date:

2020

Document Version

Peer reviewed version

[Link to publication](https://doi.org/10.1214/19-EJP397)

University of Bath

Alternative formats

If you require this document in an alternative format, please contact:
openaccess@bath.ac.uk

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

LOCAL BOUNDS FOR STOCHASTIC REACTION DIFFUSION EQUATIONS

AUGUSTIN MOINAT AND HENDRIK WEBER

ABSTRACT. We prove a priori bounds for solutions of stochastic reaction diffusion equations with super-linear damping in the reaction term. These bounds provide a control on the supremum of solutions on any compact space-time set which only depends on the specific realisation of the noise on a slightly larger set and which holds uniformly over all possible space-time boundary values. This constitutes a space-time version of the so-called "coming down from infinity" property.

Bounds of this type are very useful to control the large scale behaviour of solutions effectively and can be used, for example, to construct solutions on the full space even if the driving noise term has no decay at infinity.

Our method shows the interplay of the large scale behaviour, dictated by the non-linearity, and the small scale oscillations, dictated by the rough driving noise. As a by-product we show that there is a close relation between the regularity of the driving noise term and the integrability of solutions.

1. INTRODUCTION

We are interested in reaction diffusion equations of the type

$$(1.1) \quad (\partial_t - \Delta)u = -f(u) + \zeta,$$

over $\mathbb{R}_t \times \mathbb{R}_x^d$ where ζ is an irregular distribution. The example we have in mind is the case where ζ is a random noise term, such as space-time white noise for $d = 1$, or a noise which is "white in time and coloured in space" for $d \geq 2$. However, we mention right away that our main result is purely deterministic and the only information about ζ that enters is its regularity measured in a suitable space of distributions. The non-linearity f is assumed to be continuous, with super-linear growth at infinity in u . We derive a priori bounds on u .

It is well-known that if f satisfies the so-called Osgood condition, that is if f satisfies $\int_1^\infty \frac{1}{f(u)} du < \infty$, then solutions of the ODE $\dot{x} = -f(x)$ "come down from infinity in finite time" (see [17]). This means that if x solves the equation over $[0, t]$, then automatically $x(t)$ satisfies a bound which depends on t , but holds uniformly over all possible choices of initial datum $x(0) > 0$. Similar statements can be derived for reaction diffusion equations based on a comparison principle (see e.g. [19, Chapter 14]) and also stochastic reaction diffusion equations (see e.g. [4, Theorem 6.2.3] and [5]). These bounds are powerful tools to study the long-time behaviour of solutions, both in the deterministic and in the stochastic setting - see e.g. [20] for a construction of invariant measures for stochastic PDEs based on such bounds.

Our main result, Theorem 3.1, is a space-time version of such a bound for solutions of (1.1) with $f(u) = u|u|^{m-1} + g(u)$ where g is bounded. We consider a continuous functions $u: \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ and we assume that (1.1) holds for (t, x) in a

HW is supported by the Royal Society through the University Research Fellowship UF140187.

cylinder¹, say

$$P_0 := (0, 1) \times (-1, 1)^d.$$

Then for $R < \frac{1}{2}$ the L^∞ norm $\|u\|_{P_R}$ of u on a smaller cylinder

$$(1.2) \quad P_R := (R^2, 1) \times (-(1-R), 1-R)^d,$$

satisfies a bound which only depends on R and a distributional norm of ζ restricted to the original cylinder P_0 :

$$(1.3) \quad \|u\|_{P_R} \leq C(\alpha, d, m) \max \left\{ R^{-\frac{2}{m-1}}, [\zeta]_{\alpha-2, P_0}^{\frac{2}{2+(m-1)\alpha}}, \|g\|^{\frac{1}{m}} \right\},$$

where $[\zeta]_{\alpha-2, P_0}$ the space-time Hölder norm of order $\alpha - 2$ on P_0 (see (3.8) below for a precise definition), and $\|g\|$ refers to the supremum norm of g .

One possible application of the bound (1.3) is the *construction* of solutions to (1.1) on the full space. The standard approach to solve stochastic reaction diffusion equations [21, 10, 7] consists of writing the equation in its mild form and solving the corresponding fixed point problem using Picard iterations. However, this approach requires a pathwise uniform-in- x control on ζ , which typically only holds on bounded domains or if ζ decays at ∞ ; the interesting case of spatially stationary noise cannot be treated directly in this way. This problem was overcome in [12] where solutions were first constructed on a sequence of growing tori and then a compactness argument in a space with weights was used to pass to the limit. The strong localisation obtained in (1.3) should allow to significantly simplify this construction.

The estimate (1.3) also has an interesting consequence for the integrability of u . In fact, we are mostly interested in the case where ζ is a random distribution with Gaussian tails such that $\mathbb{E}[e^{\epsilon[\zeta]_{\alpha-2, P_0}^2}]$ is finite for $\epsilon > 0$ small enough. The estimate (1.3) then immediately implies that for any $R > 0$ and for $\epsilon > 0$ small enough we get $\mathbb{E}[e^{\epsilon\|u\|_{P_R}^{2+(m-1)\alpha}}] < \infty$. So $\|u\|_{P_R}$ has lighter tails than Gaussian. We observe that better pathwise *regularity* for ζ leads to better *integrability* with respect to the probability distribution for u . In the special case of one-dimensional reaction-diffusion equations where ζ is a space-time white noise, equation (1.1) equipped with suitable boundary conditions defines a reversible Markov process, and an explicit expression of the equilibrium measure is available. In Section 7 we argue that in this case the integrability we derive from estimate (1.3) coincides with the integrability derived from the explicit invariant measure.

Finally, our method offers a new perspective on singular SPDE. Our starting point is Hairer's notion [11] of *subcriticality* which in the context of (1.1) states, roughly speaking, that the *small scale* behaviour of solutions should be determined by the interplay of the heat operator and the rough driving noise ζ , while on *large scales* the non-linearity becomes dominant. We implement this philosophy by regularising (1.1) on a scale T by convolving the equation with a suitable regularising kernel, arriving at

$$(1.4) \quad (\partial_t - \Delta)u_T = -u_T|u_T|^{m-1} + g(u)_T + \zeta_T + [u_T|u_T|^{m-1} - (u|u|^{m-1})_T],$$

where the subscript T denotes a regularised quantity. The extra term $[u_T|u_T|^{m-1} - (u|u|^{m-1})_T]$ on the right hand side appears because regularisation and application of the polynomial do not commute. We then use a low regularity version of classical Schauder theory, Lemma 5.1, to control the error term $[u_T|u_T|^{m-1} - (u|u|^{m-1})_T]$.

¹Of course the equation (1.1) has to be interpreted in a distributional sense, so this condition means that it holds when tested against smooth test-functions which are supported in the cylinder, see (3.1).

Using this bound, the remaining terms can be treated as in the smooth case (see Theorem 4.4).

The theory of regularity structures is indeed a main motivation for this work. A priori including the "coming-down from infinity" property have been proven for singular SPDEs, namely the dynamic ϕ_2^{2m} [16, 20] and ϕ_3^4 models [15, 2, 9] both on compact domains and on the full space. The works on ϕ_3^4 all relied on Fourier methods, the method of paracontrolled distributions, rather than the theory of regularity structures. The bounds obtained there imply coming down from infinity in time only, in the case of ϕ_3^4 on the full space [9] in a weighted space. The ideas presented here can be extended to these more singular equations when the low regularity Schauder estimate, Lemma 5.1, is replaced by a suitable version of the Schauder estimates from the theory of regularity structures. This is the content of our companion paper [14]. There we show that our method significantly simplifies the technical arguments used in [15, 2, 9] and extend its scope to construct solutions on the full space without the need for weights.

In the more regular case presented here it would be natural to aim to also include more general non-linearities, such as functions with faster than polynomial growth (e.g. the $\exp(\phi)$ model $f(u) = \sinh(u)$, see [1]) or functions of slower than polynomial growth such as $f(u) \sim u \log(u)^\delta$ for $\delta > 2$. In this case the commutator term arising in (1.4) turns into

$$f(u_T) - f(u)_T.$$

Unfortunately, our method crucially on the fact that $xf'(x) \lesssim f(x)$ which holds for polynomial f , but not for functions with exponential growth. Also, another part of our argument excludes functions that grow too slowly (in the proof of Theorem 3.1 we need to sum $\Theta(u) := \frac{f(u)}{u}$ for $u = 2^{-k}$, $k \in \mathbb{N}$), thus essentially restricting us to polynomial f . However, in the case of a more general non-linearity f , we implement a more standard argument based on subtracting the solution w to the linear equation

$$(\partial_t - \Delta)w = \zeta,$$

and we do not pass through the regularised equation (1.4). We then get the property of "coming down from infinity" for the remainder $u - w$ in Corollary 4.6. For example, when $f(u) = \sinh(u)$, the strong damping implies that u "comes down from infinity" much more quickly than in the polynomial case – in this case the function $R^{-\frac{2}{m-1}}$ in (1.3) turns into $\Theta^{-1}(R^{-2})$, where $\Theta(R) = \frac{\sinh(R)}{R}$. For very weak damping $f(u) \sim u \log(u)^{\alpha+2}$, we obtain a slow coming down from infinity, of order $\exp(R^{-\frac{2}{\alpha}})$. In fact, this method is even easier than the method we use for the polynomial case, but it has two significant disadvantages: On the one hand, it is impossible to measure the fine interplay between regularity of ζ and integrability of u in this way, because the remainder $u - w$ can never have better integrability than the Gaussian process w . More importantly, the more sophisticated method we use in the proof of our main theorem is crucial when dealing with more singular equations [14].

The rest of the paper is structured as follows: In Section 2 we discuss the elementary case of the stochastic ODE

$$dx(t) = -|x(t)|^{m-1}x(t)dt + dw(t),$$

in which our strategy and also the interplay between the regularity of the noise and integrability of the solution becomes apparent in a technically simple context. In Section 3 we introduce the framework and state the main result. The proof is split into Sections 4–5: In Section 4 we present a proof of the "space-time coming down

from infinity” in the case where ζ is replaced by a smooth function. The argument relies on a maximum principle. As a Corollary, as discussed above, we derive the bounds on the remainder $u - w$ in the case of general, not necessarily polynomial f . In Section 5 the result of Section 4 is applied to the regularised equation 1.4 and combined with Schauder estimates to bound the commutator concluding the proof of our main result. In Section 6 we discuss the case of a random distribution ζ given by the time-derivative of the stochastic integral $\int_0^t \sigma dW$ for an adapted bounded process $\sigma = \sigma(s, x)$ and a distribution valued Wiener process W with suitable (spatial) covariance operator. We show Gaussian estimates for $[\zeta]_{\alpha-2}$ and thus better than Gaussian bounds for u . Finally, in the special case of space-time white noise in one spatial dimension we show that the integrability obtained from our method coincides with the integrability of the process in equilibrium obtained from the explicit invariant measure.

2. THE ODE CASE

Before dealing with equation (1.1) we briefly discuss the case of a (stochastic) ordinary differential equation

$$(2.1) \quad dx(t) = -|x(t)|^{m-1}x(t)dt + dw(t)$$

for a standard Brownian motion $w(t)$ and for $m > 1$. It is well known that (2.1) defines a reversible Markov process with respect to the measure

$$(2.2) \quad \mu(dx) \propto \exp\left(-\frac{2}{m+1}|x|^{m+1}\right)dx.$$

We seek to derive optimal bounds on solutions of $x(t)$ directly from the equation (2.1).

As a starting point, consider the case of an ordinary differential equation driven by a regular noise term η

$$(2.3) \quad \dot{x}(t) = -x(t)|x(t)|^{m-1} + \eta(t).$$

A simple ODE comparison Lemma, see [20, Lemma 3.8], shows that for $t \in (0, 1]$

$$|x(t)| \leq C(m) \max\left\{t^{-\frac{1}{m-1}}, \left(\sup_{t \in [0,1]} |\eta(t)|\right)^{\frac{1}{m}}\right\},$$

uniformly over all choices of initial datum $x(0)$. If η is a Gaussian process, such that the random variable $\sup_{t \in [0,1]} |\eta(t)|$ has finite Gaussian moments, this bound implies that for $\epsilon > 0$ small enough

$$(2.4) \quad \mathbb{E} [\exp(\epsilon |x(1)|^{2m})] < \infty.$$

In particular, in this regular case we get much better integrability than under the measure (2.2). The following deterministic lemma shows that the difference in integrability is closely related to the regularity of the driving signal.

Lemma 2.1. *Let $w: [0, 1] \rightarrow \mathbb{R}$ be α -Hölder continuous for some $\alpha \in (0, 1)$ with $w(0) = 0$. For some $m > 1$ let $x: [0, 1] \rightarrow \mathbb{R}$ be a continuous solution to*

$$(2.5) \quad x(t) = x(0) - \int_0^t |x(s)|^{m-1}x(s)ds + w(t).$$

Then for $t \in (0, 1]$

$$(2.6) \quad |x(t)| \lesssim \max\left\{t^{-\frac{1}{m-1}}, [w]_{\alpha}^{\frac{1}{1+(m-1)\alpha}}\right\},$$

where $[w]_{\alpha} = \sup_{0 \leq s < t \leq 1} \frac{|w(t) - w(s)|}{|t - s|^{\alpha}}$ denotes the α -Hölder semi-norm. Here and in the proof we use the symbol \lesssim for $\leq C(\alpha, m)$.

If w is a random function for which $[w]_\alpha$ has Gaussian tails this estimate yields

$$\mathbb{E} \left[\exp \left(\epsilon |x(1)|^{2+(m-1)2\alpha} \right) \right] < \infty,$$

for ϵ small enough. In the Brownian case where $\alpha = \frac{1}{2}$ —the exponent $2+(m-1)2\alpha$ becomes $1+m$ —in line with (2.2) and as α approaches 1, the exponent becomes $2m$ in line with (2.4).

Proof of Lemma 2.1. As a first step, we regularise equation (2.5). To this end we introduce a smooth non-negative kernel $\Psi: \mathbb{R} \rightarrow \mathbb{R}$ which is compactly supported in $[0, 1]$ with $\int \Psi = 1$ and set $\Psi_T(t) = \frac{1}{T} \Psi(\frac{t}{T})$. For any function $f: (0, 1) \rightarrow \mathbb{R}$ and for $t \in (T, 1)$ we define the regularisation $f_T(t) = f * \Psi_T(t) = \int_{t-T}^t \Psi_T(t-s) f(s) ds$.

Convolving the integral equation (2.5) with Ψ_T and taking a time derivative leads to

$$(2.7) \quad \dot{x}_T(t) = -x_T(t)|x_T(t)|^{m-1} + \dot{w}_T(t) + [\cdot^m, (\cdot)_T]x(t) \quad \text{for } t \in (T, 1),$$

where we write $[\cdot^m, (\cdot)_T]x = [x_T|x_T|^{m-1} - (x|x|^{m-1})_T]$ for the commutator term on the right hand side.

Now we can apply the ODE comparison Lemma [20, Lemma 3.8] to get, for all $t \in (T, 1]$

$$(2.8) \quad |x_T(t)| \lesssim \max \left\{ (t-T)^{-\frac{1}{m-1}}, \left(\sup_{[T,1]} |\dot{w}_T| \right)^{\frac{1}{m}}, \left(\sup_{[T,1]} |[\cdot^m, (\cdot)_T]x| \right)^{\frac{1}{m}} \right\}.$$

To replace x_T by x and to bound the commutator term on the right hand side, we need information on the regularity of x . Indeed, using the fact that Ψ has integral 1, we first see for $t \in (T, 1]$,

$$(2.9) \quad \begin{aligned} |(x_T - x)(t)| &= \left| \int_{t-T}^t \Psi_T(t-s)(x(s) - x(t)) ds \right| \\ &\leq [x]_{\alpha, (t-T, t)} \int_{t-T}^t \Psi_T(t-s) |s-t|^\alpha ds \leq T^\alpha [x]_{\alpha, (t-T, t)}, \end{aligned}$$

where $[x]_{\alpha, I} = \sup_{s \neq t \in I} \frac{|x(t) - x(s)|}{|t-s|^\alpha}$ denotes the α -Hölder semi-norm of x restricted to the interval I . Similarly we establish a bound on the commutator: for $s \geq T$,

$$(2.10) \quad |[\cdot^m, (\cdot)_T]x(s)| \lesssim \|x\|_{(s-T, s)}^{m-1} T^\alpha [x]_{\alpha, (s-T, s)},$$

where $\|x\|_I$ is the supremum norm of x restricted to the interval I . To see (2.10) we first write

$$|[\cdot^m, (\cdot)_T]x(s)| = \left| \int_{s-T}^s \Psi_T(s-r)(x_T(s)|x_T(s)|^{m-1} - x(r)|x(r)|^{m-1}) dr \right|.$$

Then, using the mean value theorem and $|x_T(s)| \leq \|x\|_{(s-T, s)}$, we have

$$|x_T(s)|x_T(s)|^{m-1} - x(r)|x(r)|^{m-1}| \leq m \|x\|_{(s-T, s)}^{m-1} |x_T(s) - x(r)|.$$

Finally, using the triangle inequality in the form $|x_T(s) - x(r)| \leq |x_T(s) - x(s)| + |x(s) - x(r)| \leq 2T^\alpha [x]_{\alpha, (s-T, s)}$, we arrive at (2.10).

In order to control the $[x]_{\alpha, (s-T, s)}$ we refer to the equation (2.5) once more, this time without regularisation. In this context a ‘‘Schauder estimate’’ is trivially derived, simply by writing for $0 < t_1 < t_2 < 1$

$$\begin{aligned} |x(t_2) - x(t_1)| &= \left| \int_{t_1}^{t_2} x(s)|x(s)|^{m-1} ds + w(t_2) - w(t_1) \right| \\ &\leq |t_2 - t_1| \|x\|_{[t_1, t_2]}^m + |w(t_2) - w(t_1)|, \end{aligned}$$

which can be restated as

$$(2.11) \quad [x]_{\alpha, (s-T, s)} \leq T^{1-\alpha} \|x\|_{(s-T, s)}^m + [w]_\alpha.$$

Finally, concerning the noise term on the right hand side of (2.8) we write

$$(2.12) \quad \sup_{t \in [T, 1]} |\dot{w}_T(t)| = \sup_{t \in [T, 1]} \left| \int_{t-T}^t \dot{\Psi}_T(t-s)(w(s) - w(t)) ds \right| \lesssim T^{\alpha-1} [w]_\alpha.$$

Combining (2.8), (2.10), (2.11) and (2.12) we arrive at

$$|x_T(t)| \lesssim \max \left\{ (t-T)^{-\frac{1}{m-1}}, \left(T^{\alpha-1} [w]_\alpha \right)^{\frac{1}{m}}, \left(T \|x\|_{(t-T, t)}^{2m-1} \right)^{\frac{1}{m}}, \right. \\ \left. \left(T^\alpha \|x\|_{(t-T, t)}^{m-1} [w]_\alpha \right)^{\frac{1}{m}} \right\}, \quad t > T.$$

Combining this estimate with (2.9) and (2.11) this estimate turns into

$$|x(t)| \lesssim \max \left\{ (t-T)^{-\frac{1}{m-1}}, \left(T^{\alpha-1} [w]_\alpha \right)^{\frac{1}{m}}, \left(T \|x\|_{(t-T, t)}^{2m-1} \right)^{\frac{1}{m}}, \right. \\ \left. \left(T^\alpha \|x\|_{(t-T, t)}^{m-1} [w]_\alpha \right)^{\frac{1}{m}}, T \|x\|^m, T^{1-\alpha} [w]_\alpha \right\}, \quad t > T.$$

Now we choose $T = \frac{\mu}{\|x\|_{(0,1)}^{\frac{1}{m-1}}}$ for $\mu = \mu(\alpha, m) > 0$ small enough and consider t satisfying $(t-T)^{-\frac{1}{m-1}} \leq \frac{1}{2} \|x\|_{(0,1)}$. Then applying the elementary inequality $xy \leq \delta x^p + C(\delta) y^{p'}$ for $\delta > 0$ and $p, p' \in (0, 1)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ multiple times yields

$$(2.13) \quad \|x(s)\|_{(2^{m-1}+\mu)\|x\|_{(0,1)}^{1-m}, 1)} \leq \max \left\{ \frac{1}{2} \|x\|_{(0,1)}, C[w]_\alpha^{\frac{1}{1+(m-1)\alpha}} \right\},$$

for some constant $C = C(\alpha, m)$. Note that we can assume that $(2^{m-1}+\mu)\|x\|_{(0,1)}^{1-m} < 1$, because else we trivially have a bound on $\|x\|_{(0,1)}$.

We now define a finite set $0 = t_0 < \dots < t_N = 1$ by setting $t_{n+1} - t_n = (2^{m-1} + \mu)\|x\|_{(t_n, 1)}^{1-m}$ as long as the times t_{n+1} defined this way stay strictly less than 1. We terminate the sequence, once this algorithm would produce a $t_{n+1} \geq 1$ in which case we set $t_{n+1} = t_N = 1$. Note that $(2^{m-1} + \mu)\|x\|_{(t_n, 1)}^{1-m}$ is increasing in n so the sequence necessarily terminates after finitely many steps.

Applying (2.13) to the equation restarted at the times t_n we obtain for $n \leq N-1$

$$(2.14) \quad \|x(s)\|_{(t_{n+1}, 1)} \leq \max \left\{ \frac{1}{2} \|x\|_{(t_n, 1)}, C[w]_\alpha^{\frac{1}{1+(m-1)\alpha}} \right\}.$$

We now show that the estimate (2.6) holds for the points t_n for $n < N$. When the maximum in (2.14) is realised by $C[w]_\alpha^{\frac{1}{1+(m-1)\alpha}}$, then this is obvious. Else, we have for $k \leq n$, $\|x\|_{(t_n, 1)} \leq \|x\|_{(t_k, 1)} 2^{k-n}$ and hence

$$(2.15) \quad t_n = \sum_{k=0}^{n-1} t_{k+1} - t_k = (2^{m-1} + \mu) \sum_{k=0}^{n-1} \|x\|_{(t_k, 1)}^{1-m} \\ \leq (2^{m-1} + \mu) \|x\|_{(t_n, 1)}^{1-m} \sum_{k=0}^{n-1} 2^{(n-k)(1-m)} \lesssim \|x\|_{(t_n, 1)}^{1-m}.$$

For the end point t_N we have either $t_{N-1} \geq \frac{1}{2}$ or $t_N - t_{N-1} \geq \frac{1}{2}$. In the first case we invoke (2.15) for $n = N-1$ and in the second case the definition of $t_{n+1} - t_n$, in both cases yielding the existence of a constant C such that

$$\|x\|_{(t_{N-1}, 1)}^{1-m} \geq C \Rightarrow \|x\|_{(t_N, 1)} \leq C^{\frac{1}{1-m}},$$

yielding in particular a bound on $x(t_N) = x(1)$.

Finally for points $t \in (t_n, t_{n+1})$, we use the definition of $t_{n+1} - t_n$:

$$\begin{aligned} t &\leq t_{n+1} = t_{n+1} - t_n + t_n \\ &\leq (2^{m-1} + \mu) \|x\|_{(t_n, 1)}^{1-m} + t_n \lesssim \|x\|_{(t_n, 1)}^{1-m} \\ &\leq |x(t)|^{1-m}. \end{aligned}$$

□

3. SETTING AND MAIN RESULT

After this short interlude, we now go back to the parabolic equation (1.1). Throughout the rest of the paper we will say a continuous function u satisfies (1.1) on an open set $B \subseteq \mathbb{R}_t \times \mathbb{R}_x^d$ if for all smooth functions η which are supported in B we have

$$(3.1) \quad \int \int u(-\partial_t - \Delta)\eta = - \int f(u)\eta + \int \zeta\eta,$$

where the last integral $\int \zeta\eta$ should be interpreted as the duality pairing between a distribution and a test function. As usual when dealing with parabolic equations, regularity will be measured with respect to the metric

$$(3.2) \quad d((t, x), (\bar{t}, \bar{x})) = \max \left\{ |x - \bar{x}|, \sqrt{|t - \bar{t}|} \right\},$$

where $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^d . We introduce the parabolic ball of center $z = (x, t)$ and radius R in this metric d , looking only into the past:

$$(3.3) \quad B(z, R) = \{\bar{z} = (\bar{t}, \bar{x}) \in \mathbb{R} \times \mathbb{R}^d, d(z, \bar{z}) < R, \bar{t} < t\}.$$

Recall that P_R is the cylinder at distance R from P_0 , as introduced in (1.2). Note that for $R' < R$ we have $P_{R'} + B(0, R' - R) \subset P_R$.

For $\alpha \in (0, 1)$, we define the Hölder semi-norm $[\cdot]_\alpha$

$$(3.4) \quad [u]_\alpha := \sup_{z \neq \bar{z} \in \mathbb{R} \times \mathbb{R}^d} \frac{|u(z) - u(\bar{z})|}{d(z, \bar{z})^\alpha}.$$

We will often deal with local quantities: If $B \subset \mathbb{R} \times \mathbb{R}^d$ is a bounded set, then we define the local α -Hölder semi-norm $[\cdot]_{\alpha, B}$ as in (3.4) with the supremum restricted to $z, \bar{z} \in B$. Similarly, $\|\cdot\|$ denotes the supremum norm on the whole space $\mathbb{R} \times \mathbb{R}^d$ and $\|\cdot\|_B$ the supremum norm over B .

To measure distributions in negative Hölder spaces, we introduce a family of mollification operators $\{(\cdot)_T\}$ which are consistent with the scaling given by the heat operator $(x, t) = (l\bar{x}, l^2\bar{t})$. For this we fix a non-negative smooth function Ψ with support in $-B(0, 1)$ with $\Psi(z) \in [0, 1]$ for all z and with integral 1 and for $T \in (0, 1]$ set $\Psi_T(x, t) = \frac{1}{T^{d+2}} \Psi\left(\frac{x}{T}, \frac{t}{T^2}\right)$. We define the operator $(\cdot)_T$ by convolution with Ψ_T , noting that for any T , $(\cdot)_T$ is a contraction on with respect to $\|\cdot\|$. We wish to keep track of the support of the relevant functions. Since Ψ_T is compactly support in $-B(0, T)$,

$$(3.5) \quad \|h_T\|_C \leq \|h\|_{C+B(0, T)}$$

for any bounded set C . Furthermore, we mention the estimate

$$(3.6) \quad \int |\Psi_T(x - y)| d(x, y)^\alpha dy \leq T^\alpha,$$

which, as in (2.9) above, immediately implies that for any $h \in C^\alpha$, and for any bounded set C , we have

$$(3.7) \quad \|h_T - h\|_C \leq T^\alpha \sup_{z \in C} [h]_{\alpha, B(z, T)}.$$

Finally, we define the local $C^{\alpha-2}$ semi-norm of a distribution ζ for $\alpha - 2 < 0$ as

$$(3.8) \quad [\zeta]_{\alpha-2,C} = \sup_{T \leq 1} \|(\zeta)_T\|_C T^{2-\alpha}.$$

This is a localised version of the Besov norm of $B_{\infty,\infty}^{\alpha-2}$ as defined, for example in [3, Theorem 2.34]. Note that, $[\zeta]_{\alpha-2,C}$ depends only on the behaviour of the distribution ζ on the set $C + B(0,1)$ (i.e. if ζ and $\tilde{\zeta}$ coincide when tested against test-functions supported in this set, then $[\zeta - \tilde{\zeta}]_{\alpha-2,C} = 0$). Multiplication with a smooth function is a continuous operation with respect to this norm. We have for any smooth and compactly supported function η

$$(3.9) \quad [\eta\zeta]_{\alpha-2} \leq C(\eta)[\zeta]_{\alpha-2,\text{supp}(\eta)}.$$

Estimates of this type are classical and are typically proved by choosing a convenient mollifying kernel Ψ_T , see e.g. [18] for estimates based on kernels Ψ_T satisfying a semi-group property in T , or [3, Section 2.4] for a proof in the language of Littlewood-Paley theory. We refer to [18, Lemma A3] for a proof that norms defined for different kernels are equivalent. More complicated bounds of this type are also essential in our companion paper [14] and are discussed there at length.

We now state our main result, to be proven in Section 5.

Theorem 3.1. *Assume that $f(u) = u|u|^{m-1} + g(u)$ with $m \geq 1$, g bounded and ζ is of regularity $\alpha - 2$ for some $\alpha > 0$ in the sense of (3.8). There exists a constant $C = C(\alpha, m, d)$ such that if u is continuous and solves (1.1) on the cylinder P_0 in the sense of (3.1) then for all $R \in (0, \frac{1}{2})$,*

$$(3.10) \quad \|u\|_{P_R} \leq C \max \left\{ R^{-\frac{2}{m-1}}, [\zeta]_{\alpha-2,P_0}^{\frac{1}{1+(m-1)\frac{d}{2}}}, \|g\|^{\frac{1}{m}} \right\}.$$

4. MAXIMUM PRINCIPLE

4.1. Assumptions and statement. We prove a space-time version of “coming down from infinity” when there is no distribution of negative regularity involved, but we allow for a more general non-linearity. Let u be a C^2 function defined for $z \in \mathbb{R} \times \mathbb{R}^d$, for which the following holds point-wise in P_0 :

$$(4.1) \quad (\partial_t - \Delta)u(z) = -f(u, z) + g(u, z).$$

Assumption 4.1. *We make the following assumptions on f and g :*

- (1) g is a bounded function;
- (2) there exists an antisymmetric C^2 function which we also denote f such that for all z , for all $u > 0$, $f(u, z) \geq f(u)$ and $-f(-u, z) \geq f(u)$;
- (3) $f''(u) \geq 0$ for $u > 0$;
- (4) there exists a constant $c > 1$ such that $uf'(u) \geq cf(u)$.

Define $\Theta(u) = \frac{f(u)}{u}$. By (4), Θ is increasing.

Theorem 4.2. *Let $u \in C^\infty$ solve (4.1) for functions f and g satisfying Assumption 4.1. There exist $\lambda = \lambda(d) > 0$ and $C = C(c, d)$ such that the following point-wise bound on u , holds for all $(t, x) \in (0, 1) \times (-1, 1)^d$:*

$$(4.2) \quad |u(x, t)| \leq C \max \left\{ \Theta^{-1} \left(\frac{1}{\lambda^2 \min\{t, (1-x_i)^2, (1+x_i)^2, i=1\dots d\}} \right), f^{-1}(\|g\|) \right\}.$$

Note that $\min\{t, (1-x_i)^2, (1+x_i)^2, i=1\dots d\}$ is exactly the square of the distance to the boundary of $[0, 1] \times [-1, 1]^d$ in the parabolic metric. In other words, this gives a bound on $\|u\|_{P_R}$, depending only on R .

The condition $xf'(x) \geq cf(x)$ with $c > 1$ is verified exactly for $f(u) = u|u|^{c-1}$, hence any function with at least polynomial growth is included in this theorem.

For such monomials, Θ^{-1} become $x \mapsto x^{\frac{1}{c-1}}$. For functions with faster growth, the bound is going to be even stronger. However, some functions with super-linear but not polynomial growth are not included. For example $f(u) = u \log(1+u)^\alpha$ for $\alpha > 0$. For this example, $\frac{uf'(u)}{f(u)} = 1 + \frac{u\alpha}{(1+u)\log(1+u)} \rightarrow 1$ as $u \rightarrow \infty$, so point (4) in Assumption 4.1 is violated. We can still get a result in that case, under a different set of assumptions:

Assumption 4.3. *We make the following assumptions on f and g :*

- (1) g is a bounded function;
- (2) there exists an antisymmetric C^2 function which we also denote f such that for all z , for all $u > 0$, $f(u, z) \geq f(u)$ and $-f(-u, z) \geq f(u)$;
- (3) $uf'(u) \geq f(u)$ and there exist two C^2 functions f_1 and f_2 such that $f = f_1 f_2$;
- (4) f_1 is antisymmetric, $f_1'' \geq 0$ for $u > 0$;
- (5) $f_2 \geq c > 0$ and

$$(4.3) \quad f_2(u) \geq \max \left\{ \frac{1}{\left(\frac{uf_1'(u)}{f_1(u)} - 1 \right)^2}, \frac{1}{\frac{uf_1'(u)}{f_1(u)} - 1} \right\}.$$

Define now $\Theta(u) = \frac{f_1(u)}{u}$. Θ is increasing by condition (5).

In the example where we want to take $f_1(u) = u \log(1+u)^\alpha$, one can easily check that in order to satisfy condition (5), f_2 should be $\left(\frac{1+u}{\alpha u} \right)^2 \frac{\log(1+u)^2}{\alpha}$ and hence $f(u) = \frac{(1+u)^2}{\alpha^2 u} \log(1+u)^{2+\alpha}$ and $\Theta^{-1}(x) = \exp(x^{\frac{1}{\alpha}}) - 1$.

Theorem 4.4. *Let $u \in C^\infty$ solve (4.1) for functions f and g satisfying Assumption 4.3. There exist $\lambda = \lambda(d) > 0$ and $C = C(c, d)$ such that the following point-wise bound on u , holds for all $(t, x) \in (0, 1) \times (-1, 1)^d$:*

$$(4.4) \quad |u(x, t)| \leq C \max \left\{ \Theta^{-1} \left(\frac{1}{\lambda^2 \min\{t, (1-x_i)^2, (1+x_i)^2, i=1\dots d\}} \right), f^{-1}(\|g\|) \right\}.$$

Theorem 4.2 is implied by Theorem 4.4 by choosing $f_1 = f$ and $f_2 = \frac{1}{(c-1)^2}$.

Remark 4.5. The fact that under these more general assumptions Θ is not simply defined by $f(u)/u$ but instead grows more slowly, is the reason why we do not get an equivalent of Theorem 3.1, in the case of slower than polynomial growth.

4.2. Bound on the remainder. A first corollary of this result is a “coming down from infinity” result for the singular equation (1.1) with general non linearity. In the manner of [6], we expand around the solution to the linear equation: let w be the solution to

$$(4.5) \quad (\partial_t - \Delta)w = \zeta,$$

with Dirichlet boundary conditions on P_0 . We will show (in Lemma 5.1) that $w \in L^\infty$ if $\zeta \in C^{\alpha-2}$ for $\alpha > 0$. Then define $v = u - w$. If u is a solution to

$$(\partial_t - \Delta)u(z) = -f(u, z) + g(u, z) + \zeta,$$

where we assume that f, g satisfies the Assumption 4.3 then v is a solution to

$$(4.6) \quad (\partial_t - \Delta)v(z) = -f(v+w, z) + g(v+w, z)$$

on P_0 . We now use the w -dependent decomposition $f(v+w, z) = \tilde{f}(v, z) + \tilde{g}(v, z)$ defined by

$$\tilde{f}(v, z) = \begin{cases} f(v+w, z) & \text{if } |v(z)| \geq 2|w(z)| \\ f\left(\frac{v(z)}{2}\right) & \text{else,} \end{cases}$$

and $\tilde{g}(v, z) = f(v + w, z) - \tilde{f}(v, z)$. Then, on the one hand by monotonicity of f we have $\tilde{f}(v, z) \geq f(\frac{v}{2})$ and on the other hand $\|\tilde{g}\| \leq f(3\|w\|)$. The Assumptions 4.3 are then satisfied with \tilde{f} and $g + \tilde{g}$ and we can apply Theorem 4.4 to get a bound on v , and then the triangle inequality to get bounds on u . We have

$$f^{-1}(\|g + \tilde{g}\|) \leq f^{-1}(2\|g\|) + 6\|w\|.$$

A corollary of Theorem 4.4 is then:

Corollary 4.6. *Assume $\zeta \in C^{\alpha-2}$ for some $\alpha > 0$. If u is solution to (1.1) and w is solution to (4.5), then there exists constants $C = C(c, d, \alpha)$ and $\lambda = \lambda(d)$ such that*

$$(4.7) \quad \|u\|_{P_R} \leq C \max \left\{ \Theta^{-1}((\lambda R)^{-2}), f^{-1}(2\|g\|), \|w\| \right\}.$$

Keeping in mind the motivation of stochastic PDEs, where ζ is the white noise, the drawback of the expansion around the solution to the linear equation is that the integrability of u that we get out of this result is at best the one of w . As we will see in section 7, Theorem 4.4 allows for better estimates than this.

4.3. Proof of Theorem 4.4. We only prove the bound for the positive part of u . The bound for the negative part follows by symmetry. Let η be a continuous function defined on $\mathbb{R}_+ \times [-1, 1]^d$, C^2 and strictly positive on the interior and such that $\eta = 0$ on the boundary. Either $u\eta$ attains its maximum on $[0, 1] \times [-1, 1]^d$ at some point $z_0 \in (0, 1] \times (-1, 1)^d$, or it is non-positive, in which case $u \leq 0$ in $[0, 1] \times \{|x| \leq 1\}$. Assuming this is not the case, we get that at the maximum point, $0 = \nabla(u\eta)(z_0)$, i.e.

$$(4.8) \quad \nabla u = -\frac{\nabla \eta}{\eta} u.$$

If $z_0 \in \{1\} \times (-1, 1)^d$, then $\partial_t u(z_0) \geq 0$. Else, $\partial_t u(z_0) = 0$. Additionally, $\Delta u(z_0) \leq 0$ and therefore at the maximum we have

$$\begin{aligned} 0 &\leq (\partial_t - \Delta)(u\eta) = \eta(\partial_t - \Delta)u + u(\partial_t - \Delta)\eta - 2\nabla u \cdot \nabla \eta \\ &\stackrel{(4.1);(4.8)}{=} -\eta(f(u, z) - g(u, z)) + u\left((\partial_t - \Delta)\eta + 2\frac{|\nabla \eta|^2}{\eta}\right). \end{aligned}$$

Assume η satisfies the following inequality:

$$(4.9) \quad \frac{(\partial_t - \Delta)\eta}{\eta} + 2\frac{|\nabla \eta|^2}{\eta^2} \leq \frac{\eta}{2} f\left(\frac{1}{\eta}\right).$$

Then we get

$$(4.10) \quad \frac{f(u)}{u} \leq \frac{\eta}{2} f\left(\frac{1}{\eta}\right) + \frac{\|g\|}{u} \leq 2 \max \left\{ \frac{\eta}{2} f\left(\frac{1}{\eta}\right), \frac{\|g\|}{u} \right\}.$$

If the maximum is realised by the first term, then $\frac{f(u)}{u} \leq \eta f(\frac{1}{\eta})$. Since $u f'(u) \geq f(u)$, $u \mapsto \frac{f(u)}{u}$ is increasing, we have that at z_0 , $u\eta \leq 1$. If the maximum is realised by the second term, then it has to be bigger than the first one :

$$\frac{\eta}{2} f\left(\frac{1}{\eta}\right) \leq \frac{\|g\|}{u} \Rightarrow u\eta \leq 2 \frac{\|g\|}{f(\frac{1}{\eta})}.$$

We then have that at z_0 , $u\eta \leq 2$ under the condition

$$(4.11) \quad \eta \leq \frac{1}{f^{-1}(\|g\|)}.$$

In both cases, we obtain that $u \leq \frac{2}{\eta}$ on all of $[0, 1] \times \{|x| \leq 1\}$. With a choice of η satisfying the inequalities (4.9) and (4.11), we obtain good bounds on the function

u . We choose the following for $z = (t, x) \in (0, \infty) \times (-1, 1)$, for some value λ to be defined:

(4.12)

$$\eta(x, t) = \frac{1}{\Theta^{-1}(\frac{1}{\lambda^2 t}) + \sum_{i=1}^d \left(\Theta^{-1}(\frac{1}{\lambda^2(1+x_i)^2}) + \Theta^{-1}(\frac{1}{\lambda^2(1-x_i)^2}) \right) + f^{-1}(\|g\|)},$$

and we continuously extend with the value 0 on the boundary of the domain. This choice of η guarantees a bound on u that is related to the distance from the boundary of $[0, 1] \times [-1, 1]^d$, independently of the boundary conditions. Indeed,

$$(4.13) \quad \begin{aligned} & (2d+1)\Theta^{-1}\left(\frac{1}{\lambda^2 \min_i \{t, (1+x_i)^2, (1-x_i)^2\}}\right) \\ & \geq \frac{1}{\eta} - f^{-1}(\|g\|) \geq \Theta^{-1}\left(\frac{1}{\lambda^2 \min_i \{t, (1+x_i)^2, (1-x_i)^2\}}\right). \end{aligned}$$

It also satisfies $0 \leq \eta \leq \frac{1}{f^{-1}(\|g\|)}$. We will now check (4.9):

$$\partial_t \eta = \frac{\lambda^2}{(\lambda^2 t)^2} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2 t})} \eta^2.$$

We use $v = \Theta^{-1}(\frac{1}{\lambda^2 t}) \leq \frac{1}{\eta}$ and $f = f_1 f_2$. Given that $\Theta'(y) = \frac{f'_1(y)}{y} - \frac{f_1(y)}{y^2}$, we get

$$\frac{\partial_t \eta}{\eta^2 f(\frac{1}{\eta})} \leq \frac{\lambda^2}{f(\frac{1}{\eta})} \frac{\Theta(v)^2}{\Theta'(v)} \leq \frac{\lambda^2}{f_1(v)f_2(v)} \frac{\Theta(v)^2}{\Theta'(v)} = \frac{\lambda^2}{f_2(v)} \frac{1}{\frac{v f'_1(v)}{f_1(v)} - 1}.$$

Applying the condition (4.3) gives a bound on this, independent of v . We now consider the spatial derivatives.

$$\begin{aligned} \partial_i \eta &= \frac{1}{\lambda^2} \left(\frac{2}{(1+x_i)^3} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2(1+x_i)^2})} - \frac{2}{(1-x_i)^3} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2(1-x_i)^2})} \right) \eta^2. \\ \partial_i^2 \eta &= -\frac{1}{\lambda^2} \left(\frac{6}{(1+x_i)^4} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2(1+x_i)^2})} + \frac{6}{(1-x_i)^4} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2(1-x_i)^2})} \right) \eta^2 \\ &+ \frac{1}{\lambda^4} \left(\frac{4}{(1+x_i)^6} \frac{\Theta'' \circ \Theta^{-1}(\frac{1}{\lambda^2(1+x_i)^2})}{(\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2(1+x_i)^2}))^3} + \frac{4}{(1-x_i)^6} \frac{\Theta'' \circ \Theta^{-1}(\frac{1}{\lambda^2(1-x_i)^2})}{(\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2(1-x_i)^2}))^3} \right) \eta^2 \\ &+ \frac{2}{\lambda^4} \left(\frac{2}{(1+x_i)^3} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2(1+x_i)^2})} - \frac{2}{(1-x_i)^3} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2(1-x_i)^2})} \right)^2 \eta^3. \end{aligned}$$

Note that the last line is equal to $\frac{\partial_i \eta}{\eta^2} 2\eta \partial_i \eta = 2 \frac{(\partial_i \eta)^2}{\eta}$, hence it will cancel when computing $-\partial_i^2 \eta + 2 \frac{(\partial_i \eta)^2}{\eta}$. For the remaining terms, we use $v_{1,i} = \Theta^{-1}(\frac{1}{\lambda^2(1+x_i)^2})$ and $v_{2,i} = \Theta^{-1}(\frac{1}{\lambda^2(1-x_i)^2})$ and we get:

$$\begin{aligned} \frac{1}{\lambda^2 f(\frac{1}{\eta})} \left(-\frac{\partial_i^2 \eta}{\eta^2} + 2 \frac{(\partial_i \eta)^2}{\eta^3} \right) &= \frac{6\Theta(v_{1,i})^2}{f(\frac{1}{\eta}) \left(\frac{f'_1(v_{1,i})}{v_{1,i}} - \frac{f_1(v_{1,i})}{v_{1,i}^2} \right)} + \frac{6\Theta(v_{2,i})^2}{f(\frac{1}{\eta}) \left(\frac{f'_1(v_{2,i})}{v_{2,i}} - \frac{f_1(v_{2,i})}{v_{2,i}^2} \right)} \\ &- \frac{4\Theta(v_{1,i})^3}{f(\frac{1}{\eta})} \frac{\frac{f''_1(v_{1,i})}{v_{1,i}} - \frac{2}{v_{1,i}} \left(\frac{f'_1(v_{1,i})}{v_{1,i}} - \frac{f_1(v_{1,i})}{v_{1,i}^2} \right)}{\left(\frac{f'_1(v_{1,i})}{v_{1,i}} - \frac{f_1(v_{1,i})}{v_{1,i}^2} \right)^3} \\ &- \frac{4\Theta(v_{2,i})^3}{f(\frac{1}{\eta})} \frac{\frac{f''_1(v_{2,i})}{v_{2,i}} - \frac{2}{v_{2,i}} \left(\frac{f'_1(v_{2,i})}{v_{2,i}} - \frac{f_1(v_{2,i})}{v_{2,i}^2} \right)}{\left(\frac{f'_1(v_{2,i})}{v_{2,i}} - \frac{f_1(v_{2,i})}{v_{2,i}^2} \right)^3}. \end{aligned}$$

Using that f is increasing, the bound (4.13) and $f = f_1 f_2$, we have that $f(\frac{1}{\eta}) \geq f_1(v_{j,i}) f_2(v_{j,i})$ for $j \in \{1, 2\}$. We also know that $f_2'' > 0$, hence we get:

$$\begin{aligned} \frac{1}{f(\frac{1}{\eta})} \left(-\frac{\partial_i^2 \eta}{\eta^2} + 2 \frac{(\partial_i \eta)^2}{\eta^3} \right) &\leq \frac{6\lambda^2}{f_2(v_{1,i}) \left(\frac{f_1'(v_{1,i}) v_{1,i}}{f_1(v_{1,i})} - 1 \right)} + \frac{6\lambda^2}{f_2(v_{2,i}) \left(\frac{f_1'(v_{2,i}) v_{2,i}}{f_1(v_{2,i})} - 1 \right)} \\ &\quad + \frac{8\lambda^2}{f_2(v_{1,i}) \left(\frac{f_1'(v_{1,i}) v_{1,i}}{f_1(v_{1,i})} - 1 \right)^2} + \frac{8\lambda^2}{f_2(v_{2,i}) \left(\frac{f_1'(v_{2,i}) v_{2,i}}{f_1(v_{2,i})} - 1 \right)^2}. \end{aligned}$$

We conclude this proof by using the condition (4.3) and setting $\lambda = (28d + 1)^{-\frac{1}{2}}$.

5. PROOF OF THE MAIN RESULT

5.1. Low regularity Schauder estimate. We give here a proof of a low regularity Schauder estimate in our setting.

Lemma 5.1. *Let u be compactly supported in $B(0, 1)$ and let $f := (\partial_t - \Delta)u$. Then for $\alpha > 0$ there exists a constant $C = C(\alpha, d)$ such that*

$$(5.1) \quad [u]_\alpha \leq C \sup_{T \leq 1} T^{2-\alpha} \|f_T\|.$$

Proof. Throughout the proof, \lesssim will denote a bound up to a multiplicative constant, which may change from line to line, but which always depends only on α and d . Define $N = \sup_{T \leq 1} T^{2-\alpha} \|f_T\|$. Since $(\cdot)_T$ denotes the convolution with a smooth kernel, it commutes with derivatives. We know that for $T < 1$, for any $l \in \text{span}\{1, x_i, i \in \{1, \dots, d\}\}$, we have on $\mathbb{R} \times \mathbb{R}^d$,

$$(\partial_t - \Delta)(u_T - l) = f_T.$$

For $z_0 \in B(0, 1)$, for some $L > 0$ to be fixed below, define $v_>$ as the solution to

$$(\partial_t - \Delta)v_> = \mathbf{1}_{\{B(z_0, L)\}} f_T, \quad v_>|_{\partial B(z_0, L)} = 0,$$

where $\partial B(z_0, L) = \{z = (t, x), d(z, z_0) = L, t \leq t_0\}$ is the parabolic boundary of $B(z_0, L)$. The first interesting inequality we get from standard heat equation estimates [13, Cor.8.1.5] is

$$(5.2) \quad \|v_>\| \lesssim L^2 \|f_T\| \leq L^2 T^{\alpha-2} N.$$

Define $v_< = u_T - v_>$. As $(\partial_t - \Delta)v_< = 0$ on $B(z_0, L)$ for any differential operator $D \in \{\partial_t, \partial_i \partial_j, i, j \in \{1, \dots, d\}\}$,

$$\|Dv_<\|_{B_{\frac{L}{2}}} \lesssim L^{-2} \inf_l \|u_T - l\|_{B(z_0, L)},$$

where l runs over all function spanned by 1 and $x_i, i \in \{1, \dots, d\}$. Therefore, for any $R < \frac{L}{2}$, for the same range of operator D , for a suitably chosen $l_R \in \text{span}\{1, x_i, i \in \{1, \dots, d\}\}$,

$$\|v_< - l_R\|_{B(z_0, R)} \leq R^2 \|Dv_<\|_{B(z_0, R)} \lesssim \left(\frac{R}{L}\right)^2 \inf_l \|u_T - l\|_{B(z_0, L)}.$$

Using the definition of $v_<$ and the triangle inequality,

$$\|u_T - l_R\|_{B(z_0, R)} - \|v_>\|_{B(z_0, R)} \lesssim \left(\frac{R}{L}\right)^2 \inf_l \|u_T - l\|_{B(z_0, L)}.$$

From (5.2),

$$(5.3) \quad \begin{aligned} \frac{1}{R^\alpha} \|u_T - l_R\| &\lesssim \left(\frac{R}{L}\right)^{2-\alpha} \frac{1}{L^\alpha} \inf_l \|u_T - l\|_{B(z_0, L)} \\ &\quad + \left(\frac{L}{T}\right)^2 \left(\frac{T}{R}\right)^\alpha N. \end{aligned}$$

Furthermore, from (3.7) we get,

$$(5.4) \quad \frac{1}{R^\alpha} \|u - l_R\|_{B(z_0, R)} \lesssim \frac{1}{R^\alpha} \|u_T - l_R\|_{B(z_0, R)} + \left(\frac{T}{R}\right)^\alpha [u]_\alpha.$$

Similarly, for any $l \in \text{span}\{1, x_i, i \in \{1, \dots, d\}\}$

$$(5.5) \quad \frac{1}{L^\alpha} \|u_T - l\|_{B(z_0, L)} \lesssim \frac{1}{L^\alpha} \|u - l\|_{B(z_0, L)} + \left(\frac{T}{L}\right)^\alpha [u]_\alpha.$$

Hence for $0 < \epsilon < 1$, for $T = \epsilon R = \epsilon^2 L$, (5.3) and (5.4), (5.5) give:

$$(5.6) \quad \frac{1}{R^\alpha} \inf_l \|u - l\|_{B(z_0, R)} \lesssim \epsilon^{2-\alpha} \frac{1}{L^\alpha} \inf_l \|u - l\|_{B(z_0, L)}$$

$$(5.7) \quad + (\epsilon^\alpha + \epsilon^{2\alpha}) [u]_\alpha + \epsilon^{\alpha-4} N.$$

Note that $[u]_\alpha \sim \sup_{z_0} \sup_L \frac{1}{L^\alpha} \inf_l \|u - l\|_{B(z_0, L)}$, hence

$$(5.8) \quad [u]_\alpha \lesssim (\epsilon^{2-\alpha} + \epsilon^\alpha + \epsilon^{2\alpha}) [u]_\alpha + \epsilon^{\alpha-4} N.$$

By making ϵ small enough, we can absorb $[u]_\alpha$ in the right hand side of (5.8) into the left hand side, concluding the proof of the Schauder estimate (5.1). Note that for this last step we needed the assumption $[u]_\alpha < \infty$. This assumption can be removed as by regularising the equation first, we have that uniformly for any $\tau > 0$,

$$[u_\tau]_\alpha \lesssim \sup_{T \leq 1} T^{2-\alpha} \|(f_\tau)_T\| \leq \sup_{T \leq 1} T^{2-\alpha} \|f_T\|,$$

and as u is continuous, we can pass to the limit for $\tau \rightarrow 0$. \square

5.2. Proof of Theorem 3.1. From now on, $f(u) = |u|^{m-1}u$. In particular, Theorem 4.2 holds with $\Theta^{-1}(R^{-2}) = R^{-\frac{1}{m-1}}$. The proof relies on two arguments. The small scale oscillations are controlled via Schauder theory and the large scale behaviour through the maximum principle derived in section 4, which applies only to regular objects. A connection between the two is established via the convolution of the equation with the kernel introduced in Section 3, which produces a commutator term. The technicality of the proof lies in balancing the contribution of the commutator and the contribution of the irregular noise.

Throughout the proof, \lesssim will denote a bound up to a multiplicative constant which may change from line to line, but will only depend on d, m and α . We will also write u^m as a short-hand for $u|u|^{m-1}$, as in the case when m is an odd integer. **STEP 1: Local Schauder estimate.** We claim that for any $R > 0$, for any $k > 2$,

$$(5.9) \quad \begin{aligned} [u]_{\alpha, B(z, R)} &\lesssim \sup_{T \leq kR} T^{2-\alpha} \|(\partial_t - \Delta)(u \mathbf{1}_{\{B(z, kR)\}})_T\| + (kR)^{-\alpha} \|u\|_{B(z, kR)} \\ &\leq (kR)^{2-\alpha} \|u\|_{B(z, kR)}^m + (kR)^{2-\alpha} \|g\| + [\zeta]_{\alpha-2, B(z, kR)} \\ &\quad + (kR)^{-\alpha} \|u\|_{B(z, kR)}. \end{aligned}$$

We prove this Schauder estimate by applying some cut-off functions and using the Lemma 5.1. By scaling and translation, it is enough to prove for some C^α function U ,

$$(5.10) \quad [U]_{\alpha, B(0, \frac{1}{2})} \lesssim \sup_{T \leq 1} T^{2-\alpha} \|(\partial_t - \Delta)(U \mathbf{1}_{\{B(0, 1)\}})_T\| + \|U\|_{B(0, 1)}.$$

Indeed, since $[u]_{\alpha, B(0, \frac{1}{k})} \leq [u]_{\alpha, B(0, \frac{1}{2})}$, if we have (5.10), define $U(t, x) = u((kR)^2(t - t_0), kR(x - x_0))$. Then

$$\begin{aligned} \|u\|_{B(z, kR)} &= \|U\|_{B(0, 1)}, \quad [u]_{\alpha, B(z, R)} = (kR)^\alpha [U]_{\alpha, B(0, \frac{1}{k})} \\ \sup_{T \leq kR} T^{2-\alpha} \|(\partial_t - \Delta)(u \mathbf{1}_{\{B(z, kR)\}})_T\| &= \sup_{T \leq 1} T^{2-\alpha} \|(\partial_t - \Delta)(U \mathbf{1}_{\{B(0, 1)\}})_T\|. \end{aligned}$$

We proceed to prove (5.10). Let η be a cut-off function, with value 1 on $B(0, \frac{1}{2})$ and 0 on $B(0, 1)^C$, and such that $\|\nabla \eta\| \leq 4$ and $\|(\Delta + \partial_t)\eta\| \leq 4$. Then

$$(5.11) \quad (\partial_t - \Delta)U\eta = \eta(\partial_t - \Delta)U + U(\partial_t + \Delta)\eta - 2\nabla \cdot (U\nabla \eta).$$

By applying Lemma 5.1 to Equation (5.11) we get that:

$$(5.12) \quad [U\eta]_\alpha \lesssim \sup_{0 < T < 1} T^{2-\alpha} \|(\eta(\partial_t - \Delta)U + U(\partial_t + \Delta)\eta - 2\nabla \cdot (U\nabla \eta))_T\|.$$

We apply the triangle inequality and make use of (3.9) to bound each of these terms as follows.

$$\|(\eta(\partial_t - \Delta)U)_T\| \lesssim \|(\partial_t - \Delta)(U\mathbf{1}_{\{B(0,1)\}})_T\|,$$

$$\|(U(\partial_t + \Delta)\eta)_T\| \leq \|U(\partial_t + \Delta)\eta\| \lesssim \|U\|_{B(0,1)},$$

$$\begin{aligned} \|(\nabla \cdot (U\nabla \eta))_T\| &= \sup_z \int (U\nabla \eta)(z - \bar{z}) \cdot \nabla \Psi_T(\bar{z}) d\bar{z} \\ &\leq \|U\nabla \eta\| \|\nabla \Psi_T\|_{L^1} \\ &\lesssim \frac{1}{T} \|U\|_{B(0,1)}. \end{aligned}$$

Since $\alpha < 1$, we have

$$\begin{aligned} \sup_{0 < T < 1} T^{2-\alpha} \|(\eta(\partial_t - \Delta)U + U(\partial_t + \Delta)\eta - 2\nabla \cdot (U\nabla \eta))_T\| \\ \lesssim \|(\partial_t - \Delta)(U\mathbf{1}_{\{B(0,1)\}})_T\| + \|U\|_{B(0,1)}. \end{aligned}$$

This concludes the proof of (5.10), hence the proof of (5.9).

STEP 2: Application of Maximum principle. We convolve the equation (1.1) with Ψ_T , where $T \in (0, 1)$ will be specified later:

$$(5.13) \quad (\partial_t - \Delta)u_T = -(u_T)^m + g_T + \zeta_T + ((u_T)^m - (u^m)_T).$$

Theorem 4.4 implies that for all $0 < R' < R < \frac{1}{2}$,

$$(5.14) \quad \|u_T\|_{P_R} \lesssim \max \left\{ \left(\frac{1}{(R - R')^2} \right)^{\frac{1}{m-1}}, \|g\|_{P_R'}^{\frac{1}{m}}, \|\zeta_T\|_{P_R'}^{\frac{1}{m}}, \left(\|(u_T)^m - (u^m)_T\|_{P_{R'}} \right)^{\frac{1}{m}} \right\}.$$

The goal is now to balance the commutator and the term with the noise. This will be done by choosing the parameter T appropriately.

STEP 3: Bounds on the commutator. We need estimates on the commutator $(u_T)^m - (u^m)_T$. This is obtained as u is C^α , using the moment bounds (3.6) and (3.7).

$$\begin{aligned} ((u_T)^m - (u^m)_T)(z) &= \int \Psi_T(z - \bar{z}) (u_T(z)^m - u(\bar{z})^m) d\bar{z} \\ &= \int \Psi_T(z - \bar{z}) \int_0^1 ((u)_T(z) - u(\bar{z})) m(\lambda(u)_T(z) + (1 - \lambda)u(\bar{z}))^{m-1} d\lambda d\bar{z} \\ &\leq m \|u\|_{B(z,T)}^{m-1} \int \Psi_T(z - \bar{z}) (u_T(z) - u(z) + u(z) - u(\bar{z})) d\bar{z} \\ &\leq m \|u\|_{B(z,T)}^{m-1} \int \Psi_T(z - \bar{z}) (T^\alpha [u]_{\alpha, B(z,T)} + [u]_{\alpha, B(z,T)} d(z, \bar{z})^\alpha) d\bar{z} \\ &\leq 2m \|u\|_{B(z,T)}^{m-1} T^\alpha [u]_{\alpha, B(z,T)}. \end{aligned}$$

Since this is true for all $z \in P_R$,

$$(5.15) \quad \|(u_T)^m - (u^m)_T\|_{P_R} \leq 2m \|u\|_{P_{R-T}}^{m-1} \sup_{z \in P_R} [u]_{\alpha, B(z, T)} T^\alpha.$$

Using the local Schauder estimate (5.9) gives, for any $k > 2$:

$$(5.16) \quad \begin{aligned} \|(u_T)^m - (u^m)_T\|_{P_R} &\lesssim T^2 k^{2-\alpha} (\|u\|_{P_{R-kT}}^{2m-1} + \|u\|_{P_{R-kT}}^{m-1} \|g\|) \\ &+ \|u\|_{P_{R-T}}^{m-1} \sup_{z \in P_R} [\zeta]_{\alpha-2, B(z, kR)} T^\alpha + k^{-\alpha} \|u\|_{P_{R-kT}}^m. \end{aligned}$$

STEP 4: Boot-strapping. We show here that for k, T such that $2(k+1)T \leq 1$, with as before $k > 2$, and for $1 \geq R \geq 2(k+1)T$, we have

$$(5.17) \quad \begin{aligned} \|u\|_{P_R} &\lesssim \max \left\{ R^{\frac{2}{1-m}}, \|g\|_{P_0}^{\frac{1}{m}}, ([\zeta]_{\alpha-2, P_0} T^{\alpha-2})^{\frac{1}{m}}, (T^2 k^{2-\alpha})^{\frac{1}{m}} \|u\|_{P_0}^{2-\frac{1}{m}}, \right. \\ &\quad (T^2 k^{2-\alpha})^{\frac{1}{m}} \|u\|_{P_0}^{1-\frac{1}{m}} \|g\|_{P_0}^{\frac{1}{m}}, (\|u\|_{P_0}^{m-1} [\zeta]_{\alpha-2, P_0} T^\alpha)^{\frac{1}{m}}, k^{-\frac{\alpha}{m}} \|u\|_{P_0} \\ &\quad \left. T^2 k^{2-\alpha} \|u\|_{P_0}^m, T^2 k^{2-\alpha} \|g\|, T^\alpha [\zeta]_{\alpha-2, P_0}, k^{-\alpha} \|u\|_{P_0} \right\}. \end{aligned}$$

We need to be careful with the sets that are concerned by the norms since our different estimates always require a bit more space. We use the bound (3.7) and the Schauder estimate (5.9):

$$(5.18) \quad \begin{aligned} \|u\|_{P_R} &\leq \|u_T\|_{P_R} + T^\alpha \sup_{z \in P_R} [u]_{\alpha, B(z, T)} \\ &\leq \|u_T\|_{P_{R+T}} + T^2 k^{2-\alpha} (\|u\|_{P_{R-kT}}^m + \|g\|) \\ (5.19) \quad &+ T^\alpha [\zeta]_{\alpha-2, P_{R-kT}} + k^{-\alpha} \|u\|_{P_{R-kT}}. \end{aligned}$$

Defining $r = kT$ allows to apply the bounds (5.14), (5.16) and (3.8). As $r \geq 0$, $P_r \subset P_0$.

$$(5.20) \quad \begin{aligned} \|u_T\|_{P_{R-T}} &\lesssim \max \left\{ (R - (k+1)T)^{\frac{2}{1-m}}, \|g\|_{P_r}^{\frac{1}{m}}, \|\zeta_T\|_{P_r}^{\frac{1}{m}}, \|(u_T)^m - (u^m)_T\|_{P_r}^{\frac{1}{m}} \right\} \\ &\lesssim \max \left\{ (R - (k+1)T)^{\frac{2}{1-m}}, \|g\|_{P_0}^{\frac{1}{m}}, ([\zeta]_{\alpha-2, P_0} T^{\alpha-2})^{\frac{1}{m}}, (T^2 k^{2-\alpha})^{\frac{1}{m}} \|u\|_{P_0}^{2-\frac{1}{m}}, \right. \\ &\quad \left. (T^2 k^{2-\alpha})^{\frac{1}{m}} \|u\|_{P_0}^{1-\frac{1}{m}} \|g\|_{P_0}^{\frac{1}{m}}, (\|u\|_{P_0}^{m-1} [\zeta]_{\alpha-2, P_0} T^\alpha)^{\frac{1}{m}}, k^{-\frac{\alpha}{m}} \|u\|_{P_0} \right\}. \end{aligned}$$

If we start with $R \geq 2(k+1)T$ then $R - (k+1)T \geq \frac{R}{2}$. Putting together (5.20) and (5.18) gives (5.17).

STEP 5: Choosing T . In order to balance the term containing ζ in (5.17), we see that we should assign the value $T = \frac{\mu^{\frac{m-1}{m-2}}}{\|u\|_{P_0}^{\frac{m-1}{m-2}}}$ for some $\mu \in (0, 1)$ to be chosen. Note also that as $\mu \in (0, 1)$, $(\mu^{\alpha-2} \vee \mu^\alpha) = \mu^{\alpha-2}$. Furthermore, we impose $\mu^2 k^{2-\alpha} \leq 1$. Consequently, (5.17) becomes

$$(5.21) \quad \begin{aligned} \|u\|_{P_R} &\lesssim \max \left\{ R^{\frac{2}{1-m}}, (1 + (\mu^2 k^{2-\alpha})^{\frac{1}{m}}) \|g\|_{P_0}^{\frac{1}{m}}, \left(\mu^{\alpha-2} \|u\|_{P_0}^{(m-1)\frac{2-\alpha}{2}} [\zeta]_{\alpha-2, P_0} \right)^{\frac{1}{m}}, \right. \\ &\quad \left((\mu^2 k^{2-\alpha} \vee k^{-\alpha})^{\frac{1}{m}} + (\mu^2 k^{2-\alpha} \vee k^{-\alpha}) \right) \|u\|_{P_0}, \\ &\quad \left. \mu^2 k^{2-\alpha} \|u\|_{P_0}^{1-m} \|g\|, \frac{\mu^\alpha}{\|u\|_{P_0}^{(m-1)\frac{2}{2}}} [\zeta]_{\alpha-2, P_0} \right\}. \end{aligned}$$

STEP 6: Identification of terms. We claim that the bound above implies that there exists a positive constants C such that:

$$(5.22) \quad \|u\|_{P_R} \leq C \max \left\{ R^{\frac{2}{1-m}}, \|g\|_{\frac{1}{m}}, [\zeta]_{\alpha-2, P_0}^{\frac{1}{1+(m-1)\frac{\alpha}{2}}}, \frac{1}{2C} \|u\|_{P_0} \right\}.$$

We need to interpolate some of the arguments of the maximum in (5.21) with arguments of our goal (5.22). The first two terms are already in the right form. For the next one, a simple interpolation inequality gives that for any $\gamma > 0$,

$$\left(\mu^{\alpha-2} \|u\|_{P_0}^{(m-1)\frac{2-\alpha}{2}} [\zeta]_{\alpha-2, P_0} \right)^{\frac{1}{m}} \lesssim \mu^{\frac{\alpha-2}{m}} \max \left\{ \gamma \|u\|_{P_0}, \gamma^{\frac{\alpha-2}{m-1+\alpha}} [\zeta]_{\alpha-2, P_0}^{\frac{1}{1+(m-1)\frac{\alpha}{2}}} \right\}.$$

The next term is also in the right form, provided one chooses first k large, and then μ small. The last two terms can not be dealt with with classical interpolation, since they involve negative powers of $\|u\|_{P_0}$. For the first one, we state that always one of the following is true, for any $\gamma > 0$:

$$\|u\|_{P_0}^{-(m-1)} \|g\| \leq \gamma \|u\|_{P_0} \text{ or } \|u\|_{P_0}^m \leq \frac{1}{\gamma} \|g\|.$$

The first case gives the last argument of our objective for γ small enough. The second case gives $\|u\|_{P_R} \leq \|u\|_{P_0} \leq (\frac{1}{\gamma} \|g\|)^{\frac{1}{m}}$. We proceed similarly for the last term. One of the following is always true:

$$\mu^\alpha \|u\|_{P_0}^{-(m-1)\frac{\alpha}{2}} [\zeta]_{\alpha-2, P_0} \leq \mu^\alpha \gamma \|u\|_{P_0} \text{ or } \|u\|_{P_0}^{1+(m-1)\frac{\alpha}{2}} \leq \frac{1}{\gamma} [\zeta]_{\alpha-2, P_0}.$$

Once again the first case gives the last argument of our objective for $\mu^\alpha \gamma$ small enough, and the second case gives $\|u\|_{P_R} \leq \|u\|_{P_0} \leq (\frac{1}{\gamma} [\zeta]_{\alpha-2, P_0})^{\frac{1}{1+(m-1)\frac{\alpha}{2}}}$. We can then choose k large, μ and γ small to get the desired constant C .

STEP 7: Iterating the result. The last argument of the maximum (5.22) is greater than the first one for all R such that

$$R \leq R_1 := \left(\frac{1}{2C} \|u\|_{P_0} \right)^{\frac{1-m}{2}},$$

Let us check that this is not in contradiction with $R_1 \geq 2(k+1)T$. By definition of T and R_1 ,

$$2(k+1)T = 2(k+1) \frac{\mu}{\|u\|_{P_0}^{\frac{m-1}{2}}} = 2(k+1) \mu (2C)^{\frac{1-m}{2}} R_1 \leq R_1 \Leftrightarrow 2(k+1) \mu (2C)^{\frac{1-m}{2}} \leq 1.$$

Since $C > 1$ and $m > 1$, it is enough to have $2(k+1)\mu \leq 1$. This can be done since μ is chosen after k .

From this point, the result (5.22) can be iterated to get bounds for smaller and smaller parabolic boxes.

$$\|u\|_{P_{(R+R_{n-1})}} \leq C \max \left\{ \frac{1}{2C} \|u\|_{P_{R_{n-1}}}, \|g\|_{\frac{1}{m}}, [\zeta]_{\alpha-2, P_0}^{\frac{1}{1+(m-1)\frac{\alpha}{2}}}, \frac{1}{R^{\frac{2}{m-1}}} \right\}.$$

Define R_n recursively by

$$(5.23) \quad R_n - R_{n-1} = \left(\frac{1}{2C} \|u\|_{P_{R_{n-1}}} \right)^{\frac{1-m}{2}} = \left(\frac{1}{2C} \frac{\|u\|_{P_0}}{2^{n-1}} \right)^{\frac{1-m}{2}}.$$

We conclude by summing those increments:

$$\begin{aligned}
 R_n &= \sum_{k=1}^n R_k - R_{k-1} = \sum_{k=1}^n \left(\frac{1}{2C} \frac{\|u\|_{P_0}}{2^{k-1}} \right)^{\frac{1-m}{2}}. \\
 (5.24) \quad &= \left(\frac{\|u\|_{P_0}}{2C} \right)^{\frac{1-m}{2}} \sum_{k=0}^{n-1} (2^{\frac{1-m}{2}})^k \lesssim \left(\frac{\|u\|_{P_0}}{2C} \right)^{\frac{1-m}{2}}
 \end{aligned}$$

The same arguments as in the proof of Lemma 2.1 conclude the proof of Theorem 3.1.

6. MULTIPLICATIVE NOISE

We present one example of equation where our result applies. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space and let $(W(t, \eta), t \geq 0, \eta \in C_0^\infty(\mathbb{R}^d))$ be a Brownian motion with spatial covariance operator K on Ω . We assume that K is given by the convolution with a function with controlled blow-up near the origin, i.e.

$$(6.1) \quad K\phi(x) = \int_{\mathbb{R}^d} K(x-x')\phi(x')dx',$$

for $K \in C^\infty(\mathbb{R}^d \setminus \{0\})$ satisfying

$$(6.2) \quad |K(x)| \leq \frac{1}{|x|^\lambda},$$

for some $\lambda < 2$. If $\lambda > 1$ and $d = 1$, we allow additionally for a Dirac mass in the origin, in which case (6.1) turns into

$$(6.3) \quad K\phi(x) = \int_{\mathbb{R}} K(x-x')\phi(x')dx' + \phi(x).$$

In other words $(W(t, \eta), t \geq 0, \eta \in C_0^\infty(\mathbb{R}^d))$ is a centred Gaussian process with covariances given either by

$$\mathbb{E}W(t, \phi)W(t', \phi') = (t \wedge t') \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x)K(x-x')\phi'(x')dxdx'$$

or in the one-dimensional case

$$\begin{aligned}
 \mathbb{E}W(t, \phi)W(t', \phi') &= (t \wedge t') \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x)K(x-x')\phi'(x')dxdx' + \int_{\mathbb{R}^d} \phi(x)\phi'(x)dx \right].
 \end{aligned}$$

Let $(\sigma(t, x), t \geq 0, x \in \mathbb{R}^d)$ be a progressively measurable process, with a deterministic L^∞ bound, without loss of generality $|\sigma(t, x)| \leq 1$. Let $u(t, x)$ be a continuous process which satisfies the SPDE

$$(6.4) \quad du = (\Delta u - f(u) + g(u))dt + \sigma dW$$

on P_0 , with f satisfying the Assumptions 4.3. More precisely, for all $\eta \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$ compactly supported in P_0 we assume that the following holds almost surely:

$$\begin{aligned}
 (6.5) \quad &\int \int u(-\partial_t - \Delta)\eta dtdx \\
 &= \int \int (-f(u, z) + g(u, z))\eta dtdx + \int \int \eta(x)\sigma(t, x)dxdW(t, x),
 \end{aligned}$$

where $\int \int \eta(x)\sigma(t, x)dxdW(t, x)$ should be interpreted as a stochastic integral, as defined in [7, Chapter 4]. The following Lemma shows that the results of our deterministic analysis are applicable to this stochastic case.

The previous results do not depend on the particular choice of convolution kernel Ψ . We apply it with $\tilde{\Psi}$ defined as

$$(6.6) \quad \tilde{\Psi} = \Psi_{\frac{1}{2}} * \Psi_{\frac{1}{2}},$$

where Ψ is as defined in Section 3. It is clear that $\tilde{\Psi}$ is still non-negative, smooth and compactly supported in $B(0, 1)$. We still write $(\cdot)_T$ for the convolution with Ψ_T but we define the $C^{\alpha-2}$ norm with respect to $\tilde{\Psi}$

$$[\zeta]_{\alpha-2, C} = \sup_{0 < T \leq 1} \|(\zeta_{\frac{T}{2}})_{\frac{T}{2}}\|_C T^{2-\alpha}$$

Lemma 6.1. *We define a family of random variables $(\zeta(\eta), \eta \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d))$ by*

$$\zeta(\eta) = \int \int \eta(x) \sigma(t, x) dx dW(t, x).$$

Then there exists a random distribution $\tilde{\zeta}$ on Ω which almost surely takes values in $C^{\alpha-2}$ for any $\alpha < \frac{2-\lambda}{2}$ and such that for $\epsilon > 0$ small enough

$$(6.7) \quad \mathbb{E} \left[\exp \left(\epsilon [\tilde{\zeta}]_{\alpha-2, P_0}^2 \right) \right] < \infty.$$

Furthermore $\tilde{\zeta}$ is a modification of ζ in the sense that for all $\eta \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$ we have almost surely

$$\tilde{\zeta}(\eta) = \zeta(\eta).$$

The following corollaries are consequences of Lemma 6.1. Using first Corollary 4.6, as well as Lemma 5.1 which provides bound on the α Hölder semi-norm of w in terms of $[\zeta]_{\alpha-2}$, which in turn controls the supremum norm using the Dirichlet boundary conditions, we get:

Corollary 6.2. *Let u solve the SPDE (6.4) in the sense of (6.5) for f and g satisfying Assumption 4.3. Define $\Theta(u) = \frac{f_1(u)}{u}$. Then there exists $\epsilon_0 = \epsilon_0(c, d, \alpha) > 0$ such that for $0 < \epsilon \leq \epsilon_0$,*

$$\mathbb{E} \left[\exp \left(\epsilon \left(\sup_{0 < R \leq \frac{1}{2}} \frac{\|u\|_{P_R}}{\Theta^{-1}((\lambda R)^{-2})} \right)^2 \right) \right] < \infty.$$

Using Theorem 3.1 in the case $f(u) = u|u|^{m-1}$, we have the more optimal estimate as follows:

Corollary 6.3. *Let u solve the SPDE (6.4) in the sense of (6.5) where $f(u, z) = u|u|^{m-1}$ and g is bounded. Then there exists $\epsilon_0 = \epsilon_0(m, d, \alpha) > 0$ such that for $0 < \epsilon \leq \epsilon_0$,*

$$\mathbb{E} \left[\exp \left(\epsilon \left(\sup_{0 < R \leq \frac{1}{2}} R^{\frac{2}{m-1}} \|u\|_{P_R} \right)^{2+(m-1)\alpha} \right) \right] < \infty.$$

The proof of Lemma 6.1 relies on the following technical lemma.

Lemma 6.4. *The supremum $\sup_{0 < T \leq 1} \|(\zeta)_T\|_{P_0}^{2p} T^{2p(2-\alpha)}$ is bounded by the supremum over dyadic T only,*

$$(6.8) \quad \sup_{0 < T \leq 1} \|\zeta_T\|_{P_0}^{2p} T^{2p(2-\alpha)} \lesssim \sup_{T=2^{-k} \leq 1} \|\zeta_T\|_{P_0+B(0,1)}^{2p} T^{2p(2-\alpha)}.$$

We give the proof of this in Appendix A.

Proof of Lemma 6.1. This Lemma is a variant of [16, Lemma 9] and we refer the reader to this Lemma for the construction of a suitable modification of ζ . Here we only show the exponential integrability bound (6.7), using a similar argument as in

[18, Lemma 4.1]. Throughout this proof, \lesssim denotes a bound up to a constant that depend only on the dimension.

In the expansion in series of the exponential, we can exchange expectation and sum:

$$\mathbb{E} \left[\exp \left(\epsilon^2 \sup_{0 < T \leq 1} \|(\zeta_{\frac{T}{2}})_{\frac{T}{2}}\|_{P_0}^2 T^{2(2-\alpha)} \right) \right] = \sum_{p=0}^{\infty} \epsilon^{2p} \frac{\mathbb{E} \left[\sup_{0 \leq T \leq 1} \|(\zeta_{\frac{T}{2}})_{\frac{T}{2}}\|_{P_0}^{2p} T^{2p(2-\alpha)} \right]}{p!}.$$

Applying Lemma 6.4, we can bound the supremum over all T by the sum over dyadic T .

$$\mathbb{E} \left[\sup_{0 < T \leq 1} \|(\zeta_{\frac{T}{2}})_{\frac{T}{2}}\|_{P_0}^{2p} T^{2p(2-\alpha)} \right] \leq \sum_{T=2^{-k} \leq \frac{1}{2}} \mathbb{E} \left[\|(\zeta_T)_T\|_{P_0+B(0,1)}^{2p} \right] T^{2p(2-\alpha)}.$$

Young's inequality implies

$$\|(\zeta_T)_T\|_{P_0+B(0,1)} \leq \|\zeta_T\|_{L^q, P_0+B(0,2)} \|\Psi_T\|_{L^{q'}},$$

where the subscript means that the L^q norm of ζ_T is taken over $P_0 + B(0, 2)$ and where $q' = \frac{q-1}{q}$. By scaling, $\|\Psi_T\|_{L^{q'}} \lesssim T^{-\frac{d+2}{q}}$. We apply this with $q = 2p$.

$$\begin{aligned} \mathbb{E} [\|(\zeta_T)_T\|^{2p}] &\lesssim \mathbb{E} \left[\|\zeta_T\|_{L^{2p}, P_0+B(0,2)}^{2p} \right] T^{-(d+2)} \\ &= \mathbb{E} \left[\int_{P_0+B(0,2)} \zeta_T(t, x)^{2p} dt dx \right] T^{-(d+2)} \\ &\lesssim \sup_{z \in P_0+B(0,2)} \mathbb{E} [\zeta_T(z)^{2p}] T^{-(d+2)}. \end{aligned}$$

We bound $\mathbb{E} [\zeta_T(z)^{2p}]$ using the boundedness of σ . Without loss of generality, we show the computation for $z = (0, 0)$. By the Burkholder-Davies-Gundy inequality,

$$\begin{aligned} \mathbb{E} [\zeta_T(0, 0)^{2p}] &= \mathbb{E} \left[\left(\int_{(0,1)} \int_{\mathbb{R}^d} \Psi_T(t, x) \sigma(t, x) dW(t, x) \right)^{2p} \right] \\ &\lesssim p^p \left(\int \int \int \Psi_T(t, x) \Psi_T(t, x') \sigma(t, x) \sigma(t, x') K(x - x') dt dx dx' \right)^p \\ &\quad + \mathbf{1}_{\{d=1, \lambda > 1\}} p^p \left(\int \int \Psi_T(t, x)^2 \sigma(t, x)^2 dt dx \right)^p \\ &\lesssim p^p (T^{-\lambda-2} + \mathbf{1}_{\{d=1, \lambda > 1\}} T^{-d-2})^p \lesssim p^p T^{-p(\lambda+2)}. \end{aligned}$$

We get that $\mathbb{E} [\|(\zeta_T)_T\|^{2p}] T^{2p(2-\alpha)} \lesssim p^p T^{p(2-2\alpha-\lambda)-(d+2)}$. Since $2 - 2\alpha - \lambda > 0$, for p large enough,

$$\sum_{T=2^{-k} \leq \frac{1}{2}} \mathbb{E} [\|(\zeta_T)_T\|^{2p}] T^{2p(2-\alpha)} \lesssim p^p \frac{1}{1 - 2^{-p(2-2\alpha-\lambda)+(d+2)}}.$$

By Stirling's formula, for p large enough,

$$\epsilon^{2p} \frac{\mathbb{E} \left[\sup_{0 \leq T \leq 1} \|(\zeta_{\frac{T}{2}})_{\frac{T}{2}}\|_{P_0}^{2p} T^{2p(2-\alpha)} \right]}{p!} \lesssim \epsilon^{2p} e^p \sqrt{p},$$

hence for $\epsilon < e^{-2}$, (6.7) is verified. \square

7. INVARIANT MEASURE AND OPTIMALITY

In this last section, we consider a special case of the SPDE considered in Section 6, namely the case of a one-dimensional reaction-diffusion equation driven by an additive space-time white noise. We aim to argue that in this case the bound obtained in Corollary 6.3 is optimal in terms of stochastic integrability.

Let $d = 1$ and let W be as in Section 6 with covariance operator $K\eta(x) = \eta(x)$. It is well-known [8, Section 11.2] that if we impose Dirichlet boundary conditions on the space-interval $[-1, 1]$, then (6.5) defines a reversible Markov process with respect to the measure

$$(7.1) \quad \frac{1}{Z} \exp \left(- \int_{-1}^1 \frac{1}{m+1} |u(x)|^{m+1} dx \right) \mu(du),$$

where μ is the law of an appropriately scaled Brownian bridge and Z is a normalisation constant. From the explicit expression (7.1) one can immediately read off that under this measure the following expectations are finite for $\alpha < \frac{1}{2}$ and ϵ small enough

$$(7.2) \quad \mathbb{E} \left[\exp \left(\epsilon \int_{-1}^1 |u|^{m+1} dx \right) \right] < \infty \text{ and } \mathbb{E} \left[\exp \left(\epsilon [u]_\alpha^2 \right) \right] < \infty.$$

The following proposition shows how to interpolate these two estimates to get optimal stochastic integrability for the supremum norm $\|u\|$.

Proposition 7.1. *If $u \in C^\alpha(-1, 1)$ and u^{m+1} is integrable, u is bounded and we have the following interpolation:*

$$(7.3) \quad \left(\frac{\|u\|_{(-1,1)}}{2} \right)^{1+\alpha(m+1)} \leq \max\{[u]_\alpha \|u\|_{m+1}^{\alpha(m+1)}, \|u\|_{m+1}^{1+\alpha(m+1)}\},$$

where $\|\cdot\|_{m+1}$ refers to the L^{m+1} norm on $[-1, 1]$.

Since $2\alpha < 1$,

$$\|u\|_{m+1}^{(m+1)\alpha} [u]_\alpha \leq \|u\|_{m+1}^{m+1} + [u]_\alpha^2.$$

Hence, (7.2) implies that for ϵ small,

$$(7.4) \quad \mathbb{E} \left[\exp \left(\epsilon \|u\|^{1+(m+1)\alpha} \right) \right] < \infty.$$

On the other hand, from Theorem 3.1 and from Corollary 6.3, we get

$$(7.5) \quad \mathbb{E} \left[\exp \left(\epsilon (2^{-\frac{2}{m-1}} \|u\|_{P_{\frac{1}{2}}})^{2+(m-1)\alpha} \right) \right] < \infty.$$

Therefore, for $\alpha \rightarrow \frac{1}{2}$, the exponents in (7.4) and (7.5) both converge to $\frac{m+3}{2}$. For one dimensional space-time white noise, the term $[\zeta]_{\alpha-2, P_0}^{\frac{1}{1+(m-1)\frac{\alpha}{2}}}$ in Theorem 3.1 is an optimal control.

APPENDIX A. PROOF OF LEMMA 6.4

This proof essentially follows [18, Lemma A.3]. By splitting the interval $(0, 1)$ into $[2^{-n}, 2^{-n+1})$ for $n \geq 1$, it is enough to prove that for all $n \geq 1$, uniformly in $\lambda \in (0, 1)$

$$(A.1) \quad \|(\zeta)_{2^{-n}(1+\lambda)}\| \lesssim 2^{-n(\alpha-2)} \sup_{T=2^{-k} \leq 1} T^{2-\alpha} \|\zeta_T\|.$$

For $m \in \mathbb{N}^{d+1}$, denote $|m| = 2m_0 + \sum_{k=1}^d m_k$ the parabolic index, and $m! = \prod_{k=0}^d m_k!$. We define $A_{n,m} = \int z^n \partial^m \Psi(z) dz$ and observe that since Ψ is compactly

supported and since $\int \Psi = 1$, we have

$$A_{n,m} = \begin{cases} 0 & \text{if } |n| \leq |m|, n \neq m, \\ (-1)^{|m|} m! & \text{if } n = m. \end{cases}$$

Hence for any $\beta > 0$, $(A_{n,m})_{|n|,|m| \leq \beta}$ is an invertible linear system. By continuity of the coefficients, for r small enough (depending on β),

$$(A.2) \quad A_{n,m}^r = \int z^n \partial^m (\Psi_r * \Psi)(z) dz$$

is also an invertible linear system. Hence, there exists coefficients $(a_m)_{|m| \leq \beta}$ such that

$$(A.3) \quad \sum_{|m| \leq \beta} a_m \int z^n \partial^m (\Psi_r * \Psi)(z) dz = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{else.} \end{cases}$$

Set $\omega^0 = \sum_{|m| \leq \beta} a_m \partial^m \Psi_r$ and $\Psi' = \omega^0 * \Psi$, then

$$(A.4) \quad \int z^n \Psi'(z) dz = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{for } 0 < |n| < \beta. \end{cases}$$

Define now for $\theta > 0$ inductively $\omega^{(0,\lambda)} = \Psi_{1+\lambda}$ and $\omega^{(k+1,\lambda)} = \omega^{(k,\lambda)} - \Psi'_{\theta^k} * \omega^{(k,\lambda)}$, in order to get

$$(A.5) \quad \Psi_{1+\lambda} = \sum_{k=0}^{\infty} \Psi'_{\theta^k} * \omega^{(k,\lambda)} = \sum_{k=0}^{\infty} \Psi_{\theta^k} * \omega^{(k,\lambda)} * \omega^0.$$

Similar to (3.7) on can see that $\int |\omega^{(k,\lambda)} - \Psi'_{\theta^k} * \omega^{(k,\lambda)}| \lesssim \theta^{(\beta+1)k} \int |\partial^\beta \omega^{(k,\lambda)}|$ and $\int |\partial^\beta (\omega^{(k,\lambda)} - \Psi'_{\theta^k} * \omega^{(k,\lambda)})| \lesssim \int |\partial^\beta \omega^{(k,\lambda)}|$, hence by induction

$$(A.6) \quad \int |\omega^{(k,\lambda)}| \lesssim \theta^{(\beta+1)k} \int |\partial^\beta \Psi_{1+\lambda}| \lesssim \theta^{(\beta+1)k} \sup_{\lambda \in (0,1)} \int |\partial^\beta \Psi_{1+\lambda}|.$$

We can rescale (A.5) by $T = 2^{-k}$, and we have

$$\begin{aligned} \|(\zeta)_{2^{-n}(1+\lambda)}\|_{P_0} &= \left\| \sum_{k=0}^{\infty} \Psi_{\theta^k 2^{-n}} * \omega_{2^{-n}}^{(k,\lambda)} * \omega_{2^{-n}}^0 * \zeta \right\|_{P_0} \\ &\leq \sum_{k=0}^{\infty} \|(\zeta)_{\theta^k 2^{-n}}\|_{P_0+B(0,2^{-n})} \int |\omega_{2^{-n}}^{(k,\lambda)}| \int |\omega_{2^{-n}}^0| \\ &\leq \sup_{T=2^{-k} \leq 1} T^{2-\alpha} \|\zeta_T\|_{P_0+B(0,1)} \sup_{\lambda \in (0,1)} \int |\partial^\beta \Psi_{1+\lambda}| \int |\omega^0| \\ &\quad \sum_{k=0}^{\infty} (\theta^k 2^{-n})^{\alpha-2} \theta^{(\beta+1)k} \\ &\lesssim 2^{-n(\alpha-2)} \sup_{T=2^{-k} \leq 1} T^{2-\alpha} \|\zeta_T\|_{P_0+B(0,1)}. \end{aligned}$$

APPENDIX B. PROOF OF PROPOSITION 7.1

For any interval $I \subset [-1, 1]$,

$$|u(t) - \frac{1}{|I|} \int_I u(s) ds| \leq \frac{1}{|I|} \int_I |u(t) - u(s)| ds \leq \frac{1}{|I|} \int_I [u]_{\alpha, I} |t - s|^\alpha.$$

If $t \in I$, $|t - s|^\alpha \leq I^\alpha$. We can apply Jensen's inequality.

$$|u(t)| \leq \left(\frac{1}{|I|} \int_I |u(s)|^{m+1} ds \right)^{\frac{1}{m+1}} + |I|^\alpha [u]_{\alpha, I}.$$

And since this is true for any $I \subset [-1, 1]$, we have for any choice of $0 < x \leq 2$,

$$\|u\|_J \leq x^{-\frac{1}{m+1}} \|u\|_{m+1} + x^\alpha [u]_{\alpha, J}.$$

If $\|u\|_{m+1} \geq [u]_\alpha$ then choose $x = 1$ to get $\|u\|_{(-1,1)} \leq 2\|u\|_{m+1}$. Else choose $x = (\|u\|_{m+1}/[u]_\alpha)^{\frac{m+1}{1+\alpha(m+1)}} \leq 1$ and get $\|u\|_{(-1,1)} \leq 2[u]_\alpha^{\frac{1}{1+\alpha(m+1)}} \|u\|_{m+1}^{\frac{\alpha(m+1)}{1+\alpha(m+1)}}$. In conclusion,

$$\|u\|_{(-1,1)} \leq 2 \max\{\|u\|_{m+1}, [u]_\alpha^{\frac{1}{1+\alpha(m+1)}} \|u\|_{m+1}^{\frac{\alpha(m+1)}{1+\alpha(m+1)}}\}.$$

REFERENCES

- [1] S. Albeverio, H. Kawabi, and M. Röckner. Strong uniqueness for both dirichlet operators and stochastic dynamics to gibbs measures on a path space with exponential interactions. *Journal of Functional Analysis*, 262(2):602–638, 2012.
- [2] S. Albeverio and S. Kusuoka. The invariant measure and the flow associated to the ϕ_3^4 -quantum field model. *arXiv preprint arXiv:1711.07108*, 2017.
- [3] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343. Springer Science & Business Media, 2011.
- [4] S. Cerrai. *Second order PDE's in finite and infinite dimension: a probabilistic approach*, volume 1762. Springer Science & Business Media, 2001.
- [5] S. Cerrai. Stochastic reaction-diffusion systems with multiplicative noise and non-lipschitz reaction term. *Probability Theory and Related Fields*, 125(2):271–304, 2003.
- [6] G. Da Prato and A. Debussche. Strong solutions to the stochastic quantization equations. *The Annals of Probability*, 31(4):1900–1916, 2003.
- [7] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*. Cambridge university press, 2014.
- [8] G. Da Prato, J. Zabczyk, and J. Zabczyk. *Ergodicity for infinite dimensional systems*, volume 229. Cambridge University Press, 1996.
- [9] M. Gubinelli and M. Hofmanová. Global solutions to elliptic and parabolic ϕ^4 models in euclidean space. *arXiv preprint arXiv:1804.11253*, 2018.
- [10] M. Hairer. An introduction to stochastic pdes. *arXiv preprint arXiv:0907.4178*, 2009.
- [11] M. Hairer. A theory of regularity structures. *Inventiones mathematicae*, 198(2):269–504, 2014.
- [12] K. Iwata. An infinite dimensional stochastic differential equation with state space $C(\mathbb{R})$. *Probability theory and related fields*, 74(1):141–159, 1987.
- [13] N. V. Krylov. *Lectures on elliptic and parabolic equations in Hölder spaces*. Number 12. American Mathematical Soc., 1996.
- [14] A. Moinat and H. Weber. Tba. *TBA*, 00(0):1–13, 2018.
- [15] J.-C. Mourrat and H. Weber. The dynamic ϕ_3^4 model comes down from infinity. *Communications in Mathematical Physics*, 356(3):673–753, 2017.
- [16] J.-C. Mourrat, H. Weber, et al. Global well-posedness of the dynamic ϕ^4 model in the plane. *The Annals of Probability*, 45(4):2398–2476, 2017.
- [17] W. F. Osgood. Beweis der Existenz einer Lösung der Differentialgleichung $\frac{dy}{dx} = f(x, y)$ ohne Hinzunahme der Cauchy-Lipschitz'schen Bedingung. *Monatshefte für Mathematik und Physik*, 9(1):331–345, 1898.
- [18] F. Otto and H. Weber. Quasilinear spdes via rough paths. *arXiv preprint arXiv:1605.09744*, 2016.
- [19] J. Smoller. *Shock waves and reaction – diffusion equations*, volume 258. Springer Science & Business Media, 2012.
- [20] P. Tsatsoulis and H. Weber. Spectral gap for the stochastic quantization equation on the 2-dimensional torus. *arXiv preprint arXiv:1609.08447*, 2016.
- [21] J. B. Walsh. An introduction to stochastic partial differential equations. In *École d'Été de Probabilités de Saint Flour XIV-1984*, pages 265–439. Springer, 1986.

(Augustin Moinat) UNIVERSITY OF WARWICK, COVENTRY, UNITED KINGDOM
E-mail address: a.moinat@warwick.ac.uk

(Hendrik Weber) UNIVERSITY OF WARWICK, COVENTRY, UNITED KINGDOM
E-mail address: hendrik.weber@warwick.ac.uk